Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 2(1): 96-101
DOI:10.3934/Math.2017.1.96
Received: 3 October 2016
Accepted: 31 December 2016
Published: 10 February 2017

Research article

On the Sum of Unitary Divisors Maximum Function

Bhabesh Das[1,*] **and Helen K. Saikia**[2]

[1] Department of Mathematics, B.P.Chaliha College, Assam-781127, India
[2] Department of Mathematics, Gauhati University, Assam-781014, India

[*] **Correspondence:** Email: mtbdas99@gmail.com

Abstract: It is well-known that a positive integer d is called a unitary divisor of an integer n if $d|n$ and $\gcd\left(d, \frac{n}{d}\right) = 1$. Divisor function $\sigma^*(n)$ denote the sum of all such unitary divisors of n. In this paper we consider the maximum function $U^*(n) = \max\{k \in \mathbb{N} : \sigma^*(k)|n\}$ and study the function $U^*(n)$ for $n = p^m$, where p is a prime and $m \geq 1$.

Keywords: Unitary Divisor function; Smarandache function; Fermat prime

1. Introduction

Any function whose domain of definition is the set of positive integers is said to be an arithmetic function. Let $f : \mathbb{N} \to \mathbb{N}$ be an arithmetic function with the property that for each $n \in \mathbb{N}$ there exists at least one $k \in \mathbb{N}$ such that $n|f(k)$. Let

$$F_f(n) = \min\{k \in \mathbb{N} : n|f(k)\} \tag{1.1}$$

This function generalizes some particular functions. If $f(k) = k!$, then one gets the well known Smarandache function, while for $f(k) = \frac{k(k+1)}{2}$ one has the Pseudo Smarandache function [1, 5, 6]. The dual of these two functions are defined by J. Sandor [5, 7]. If g is an arithmetic function having the property that for each $n \in \mathbb{N}$, there exists at least one $k \in \mathbb{N}$ such that $g(k)|n$, then the dual of $F_f(n)$ is defined as

$$G_g(n) = \max\{k \in \mathbb{N} : g(k)|n\} \tag{1.2}$$

The dual Smarandache function is obtained for $g(k) = k!$ and for $g(k) = \frac{k(k+1)}{2}$ one gets the dual Pseudo-Smarandache function. The Euler minimum function has been first studied by P. Moree and H. Roskam [4] and it was independently studied by Sandor [11]. Sandor also studied the maximum and minimum functions for the various arithmetic functions like unitary toitent function $\varphi^*(n)$ [9], sum of divisors function $\sigma(n)$, product of divisors function $T(n)$ [10] , the exponential totient function $\varphi^e(n)$ [8].

2. Preliminary

A positive integer d is called a unitary divisor of n if $d|n$ and $\gcd\left(d, \frac{n}{d}\right) = 1$. The notion of unitary divisor related to arithmetical function was introduced by E.Cohen[3]. If the integer $n > 1$ has the prime factorization $n = p_1^{\alpha_1} p_2^{\alpha_2} .....p_r^{\alpha_r}$,then d is a unitary divisor of n if and only if $d = p_1^{\beta_1} p_2^{\beta_2} .....p_r^{\beta_r}$,where $\beta_i = 0$ or $\beta_i = \alpha_i$ for every $i \in \{1, 2, 3....r\}$. The unitary divisor function, denoted by $\sigma^*(n)$, is the sum of all positive unitary divisors of n. It is to noted that $\sigma^*(n)$ is a multiplicative function. Thus $\sigma^*(n)$ satisfies the functional condition $\sigma^*(nm) = \sigma^*(n)\sigma^*(m)$ for $\gcd(m, n) = 1$. If $n > 1$ has the prime factorization $n = p_1^{\alpha_1} p_2^{\alpha_2} .....p_r^{\alpha_r}$, then we have $\sigma^*(n) = \sigma^*(p_1^{\alpha_1})\sigma^*(p_2^{\alpha_2})...\sigma^*(p_r^{\alpha_r}) = (p_1^{\alpha_1} + 1)(p_2^{\alpha_2} + 1)....(p_r^{\alpha_r} + 1)$ In this paper, we consider the case (1.2) for the unitary divisor function $\sigma^*(n)$ and investigate various characteristics of this function. In (1.2), taking $g(k) = \sigma^*(k)$ we define maximum function as follows

$$U^*(n) = \max\{k \in \mathbb{N} : \sigma^*(k)|n\}$$

First we discuss some preliminary results related to the function $\sigma^*(n)$.

Lemma 2.1. Let $n \geq 2$ be a positive integer and let r denote the number of distinct prime factors of n. Then

$$\sigma^*(n) \geq (1 + n^{\frac{1}{r}})^r \geq 1 + n$$

Proof. Let $n = p_1^{\alpha_1} p_2^{\alpha_2} .....p_r^{\alpha_r}$ be the prime factorization of the natural number $n \geq 2$, where p_i are distinct primes and $\alpha_i \geq 0$. For any positive numbers $x_1, x_2, x_3..., x_r$ by Huyggens inequality,we have $((1 + x_1)(1 + x_2)(1 + x_r))^{\frac{1}{r}} \geq 1 + (x_1x_2..x_r)^{\frac{1}{r}}$ For $i = 1, 2, ..r$, putting $x_i = p_i^{\alpha_i}$ in the above inequality, we obtain $\sigma^*(n)^{\frac{1}{r}} \geq 1 + n^{\frac{1}{r}}$, giving $\sigma^*(n) \geq (1 + n^{\frac{1}{r}})^r$. Again for any numbers $a, b \geq 0, r \geq 1$, from binomial theory we have $(a + b)^r \geq a^r + b^r$. Therefore we obtain $\sigma^*(n) \geq (1 + n^{\frac{1}{r}})^r \geq 1 + n$. Thus for all $n \geq 2$, we have $\sigma^*(n) \geq 1 + n$. The equality holds only when n is a prime or n is power of a prime.

Remark 2.1. From the lemma 2.1,for all $k \geq 2$ we have $\sigma^*(k) \geq k + 1$ and from $\sigma^*(k)|n$ it follows that $\sigma^*(k) \leq n$, so $k + 1 \leq n$. Thus $U^*(n) \leq n - 1$. Therefore the maximum function $U^*(n)$ is finite and well defined.

Lemma 2.2. Let p be a prime. The equation $\sigma^*(x) = p$ has solution if and only if p is a Fermat prime.

Proof. If x is a composite number with at least two distinct prime factors, then $\sigma^*(x)$ is also a composite number. Therefore, for any composite number x with at least two distinct prime factors, $\sigma^*(x) \neq p$, a prime. So x must be of the form $x = q^\alpha$ for some prime q. Thus $x = q^\alpha$ gives $\sigma^*(x) = q^\alpha + 1 = p$ if and only if $q^\alpha = p - 1$. If $p = 2$, then $q = 1$ and $\alpha = 1$, which is impossible , so p must be an odd prime . If $p \geq 3$, then $p - 1$ is even, so we must have $q = 2$, i.e., $p = 2^\alpha + 1$. It is clear that such prime exists when α is a power of 2 giving thereby that p is Fermat prime (see [2], page-236).

Lemma 2.3. Let p be a prime. The equation $\sigma^*(x) = p^2$ only has the following two solutions: $x = 3, p = 2$ and $x = 8, p = 3$.

Proof. Let $x = p_1^{\alpha_1} p_2^{\alpha_2} .....p_r^{\alpha_r}$ be solution of $\sigma^*(x) = p^2$, then $(1 + p_1^{\alpha_1})(1 + p_2^{\alpha_2})....(1 + p_r^{\alpha_r}) = p^2$ if and only if

(a) $p_1^{\alpha_1} + 1 = p^2$
(b)$p_1^{\alpha_1} + 1 = 1$, $p_2^{\alpha_2} + 1 = p^2$
(c)$p_1^{\alpha_1} + 1 = 1$, $p_2^{\alpha_2} + 1 = p$, $p_3^{\alpha_3} + 1 = p$
(d)$p_1^{\alpha_1} + 1 = p$, $p_2^{\alpha_2} + 1 = p$

Since p_i are distinct primes, so the cases (b), (c) and (d) are impossible. Therefore only possible case is (a). If x is odd, then from the case (a) we must have only $p = 2$. In this case we have $p_1 = 3$, $\alpha_1 = 1$, so $x = 3$. If x is even then only possibility is $p_1 = 2$, so from the case (a),we have $2^{\alpha_1} = p^2 - 1$, then $2^{\alpha_1} = (p-1)(p+1)$, giving the equations $2^a = p - 1$, $2^b = p + 1$, where $a + b = \alpha_1$. Solving we get $2^b = 2(1 + 2^{a-1})$ and $p = 2^{a-1} + 2^{b-1}$. Since $2^b = 2(1 + 2^{a-1})$ is possible only when $b = 2$ and $a = 1$, therefore $p = 2^{a-1} + 2^{b-1}$ gives $p = 3$. Thus $\alpha_1 = 3$ and hence $x = 8$.

Lemma 2.4. Let p be a prime. The equation $\sigma^*(x) = p^3$ has a unique solution: $x = 7$, $p = 2$.

Proof. Proceeding as the lemma 2.3, we are to find the solution of the equation $p_1^{\alpha_1} + 1 = p^3$. If p_1 is odd, then only possible value of p is 2 and in that case the solution is $x = 7$. If p_1 is even, then $p_1 = 2$ and $2^{\alpha_1} = p^3 - 1$. In that case $p \neq 2$ and hence p is an odd prime. Since $p^3 - 1 = (p-1)(p^2 + p + 1)$ and $p^2 + p + 1$ is odd for any prime p, hence $2^{\alpha_1} \neq p^3 - 1$.

Lemma 2.5. Let $k > 1$ be an integer. The equation $\sigma^*(x) = 2^k$ is always solvable and its solutions are of the form x = Mersenne prime or x = a product of distinct Mersenne primes.

Proof. Let $x = p_1^{\alpha_1} p_2^{\alpha_2} .....p_r^{\alpha_r}$, then $(1 + p_1^{\alpha_1})(1 + p_2^{\alpha_2}) ....(1 + p_r^{\alpha_r}) = 2^k$, which gives $(1 + p_1^{\alpha_1}) = 2^{k_1}$, $(1 + p_2^{\alpha_2}) = 2^{k_2}$, ...$(1 + p_r^{\alpha_r}) = 2^{k_r}$, where $k_1 + k_2 + ... + k_r = k$. Clearly each p_i is odd. Now we consider the equation $p^\alpha = 2^a - 1$,$(a > 1)$.

If $\alpha = 2m$ is an even and $p \geq 3$, then p must be of the form $4h \pm 1$ and $p^2 = 16h \pm 8h + 1 = 8h(2h \pm 1) + 1 = 8j + 1$. Therefore $p^{2m} + 1 = (8j + 1)^m + 1 = (8r + 1) + 1 = 2(4r + 1) \neq 2^a$. If $\alpha = 2m + 1$, $(m \geq 0)$, then $p^{2m+1} + 1 = (p + 1)(p^{2m} - p^{2m-1} + ... - p + 1)$. Clearly the expression $p^{2m} - p^{2m-1} + ... - p + 1$ is odd. Thus $p^\alpha + 1 \neq 2^a$, when $\alpha = 2m + 1$,$(m > 0)$. If $m = 0$, then $p = 2^a - 1$, a prime. Any prime of the form $p = 2^a - 1$ is always a Mersenne prime. Thus each p_i is Mersenne prime. Hence the lemma is proved.

Lemma 2.6. Let p be a prime and $k > 2$ be an integer. The equation $\sigma^*(x) = p^k$ has solution only for $p = 2$.

Proof. Let $x = p_1^{\alpha_1} p_2^{\alpha_2} .....p_r^{\alpha_r}$ .Then proceeding as the lemma 2.5,we have to solve the equation of the form $q^\alpha + 1 = p^k$, where q is a prime. If q is odd , then $q^\alpha = p^k - 1$ must be odd. This is possible only when $p = 2$ and $\alpha = 1$. In that case $\sigma^*(x) = 2^k$ and by the lemma 2.5, this equation is solvable. If q is even, then only possibility is $q = 2$ and $2^\alpha = p^k - 1$ .One can easily show that this equation has no solution for $k > 2$. It is clear that p is an odd prime. If $k = 2m + 1, (m > 0)$ is an odd, then $p^{2m+1} - 1 = (p-1)(p^{2m} + p^{2m-1} + ... + p + 1)$.Since the expression $p^{2m} + p^{2m-1} + ... + p + 1$ gives an odd number, so in that case $p^{2m+1} - 1 \neq 2^\alpha$. If $k = 2m + 2, (m > 0)$ is an even (since $k > 2$), then $p^{2m+2} - 1 = 2^\alpha$. This equation gives $p^{m+1} - 1 = 2^a$ and $p^{m+1} + 1 = 2^b$,where $a + b = \alpha$. Solving we obtain $2^b - 2^a = 2$. The last equation has only solution $a = 1$,$b = 2$. Therefore we get $\alpha = 3$. For $\alpha = 3$, the equation $p^{2m+2} - 1 = 2^\alpha$ strictly implies that $m = 0$. But by our assumption $k > 2$. Hence the lemma is proved.

3. Results

In this section we discuss our main results.

Following result follows from the definition of $U^*(n)$

Theorem 3.1. For all $n \geq 1$, $\sigma^*(U^*(n)) \leq n$.

Theorem 3.2. For all $n \geq 2$, $U^*(n) \leq n - 1$

Proof. From the lemma 2.1,for all $k \geq 2$,we have $\sigma^*(k) \geq k + 1$. Putting $k = U^*(n)$, one can get $\sigma^*(U^*(n)) \geq U^*(n) + 1$. Using the theorem 3.1,we obtain $n \geq \sigma^*(U^*(n)) \geq 1 + U^*(n)$, for all $n \geq 2$.

Theorem 3.3. If p is a prime and $\alpha \geq 1$, then $U^*(p^\alpha + 1) = p^\alpha$

Proof. Since for any prime power p^α , we have $\sigma^*(p^\alpha) = p^\alpha + 1$, so we can write $\sigma^*(p^\alpha)|p^\alpha + 1$. Therefore from the definition of $U^*(n)$, we get $p^\alpha \leq U^*(p^\alpha + 1)$,for all $\alpha \geq 1$ .Putting $n = p^\alpha + 1$ in the inequality of the theorem 3.2, we get $U^*(p^\alpha + 1) \leq p^\alpha$.

Theorem 3.4. For $i = 1, 2, ...r$, let p_i be distinct primes . If n be a positive integer such that $(1 + p_1^{\alpha_1})(1 + p_2^{\alpha_2})....(1 + p_r^{\alpha_r})|n$, where $\alpha_i \geq 1$,then $U^*(n) \geq p_1^{\alpha_1} p_2^{\alpha_2} ...p_r^{\alpha_r}$

Proof. Since $\sigma^*(p_1^{\alpha_1} p_2^{\alpha_2} ...p_r^{\alpha_r}) = (1 + p_1^{\alpha_1})(1 + p_2^{\alpha_2})....(1 + p_r^{\alpha_r})|n$, so from the definition of $U^*(n)$, the result follows.

Theorem 3.5.

$$U^*(p) = \begin{cases} 2^m, & \text{if } p = 2^m + 1 \text{ is Fermat prime,} \\ 1, & \text{if } p = 2 \text{ or } p \text{ is not Fermat prime} \end{cases}$$

Proof. We have $\sigma^*(k)|p$, when $\sigma^*(k) = p$ or $\sigma^*(k) = 1$. Thus from the lemma 2.2 and the definition of $U^*(n)$ the result follows.

Theorem 3.6.

$$U^*(p^2) = \begin{cases} 3, & \text{if } p = 2, \\ 8, & \text{if } p = 3 \\ 2^m, & \text{if } p = 2^m + 1 > 3 \text{ is Fermat prime,} \\ 1, & \text{if } p \text{ is not Fermat prime} \end{cases}$$

Proof. The result follows from the lemma 2.3 and the definition of $U^*(n)$.

Theorem 3.7.

$$U^*(p^3) = \begin{cases} 7, & \text{if } p = 2, \\ 8, & \text{if } p = 3 \\ 2^m, & \text{if } p = 2^m + 1 > 3 \text{ is Fermat prime,} \\ 1, & \text{if } p \text{ is not Fermat prime} \end{cases}$$

Proof. The result follows from the lemma 2.4 and the definition of $U^*(n)$.

Theorem 3.8. $U^*(2^t) = g$, where g is the greatest product $(2^{p_1} - 1)(2^{p_2} - 1)...(2^{p_r} - 1)$ of Mersenne primes, where $p_1 + p_2 + ... + p_r \leq t$.

Proof. Let $\sigma^*(k)|2^t$,then $\sigma^*(k) = 2^a$, where $0 \leq a \leq t$ .From the definition of $U^*(n)$ and the lemma 2.5, the greatest value of such k is $k = g$, where $g = (2^{p_1} - 1)(2^{p_2} - 1)...(2^{p_r} - 1)$, with $p_1 + p_2 + ... + p_r \leq t$.

Example 3.1. For $n = 2^8$, $p_1 + p_2 + ... + p_r = 8$, so we get $p_1 = 3$, $p_2 = 5$. Therefore $g = (2^{p_1} - 1)(2^{p_2} - 1) = 217$, i.e. $U^*(2^8) = 217$.

Theorem 3.9. For $k > 3$,

$$U^*(p^k) = \begin{cases} g, & \text{if } p = 2, \text{ where } g \text{ is given in the theorem 3.8,} \\ 8, & \text{if } p = 3, \\ 2^m, & \text{if } p = 2^m + 1 > 3 \text{ is Fermat prime,} \\ 1, & \text{if } p \text{ is not Fermat prime} \end{cases}$$

Proof. The result follows from the lemma 2.6 and the definition of $U^*(n)$.

Corollary 3.10. For any $a \geq 1$, $U^*(7^a) = 1, U^*(11^a) = 1$, $U^*(13^a) = 1, U^*(19^a) = 1$ etc.

Theorem 3.11. For $a \geq 1$,any number of the form $n = (2^m + 1)(2^p - 1)^a$, $U^*(n) = 2^l$ for some l, where $2^m + 1$ is Fermat prime and $2^p - 1$ is Mersenne prime.

Proof. Since 3 is the only prime which is both Mersenne and Fermat prime, so in that case for $a \geq 1$, $n = 3^{a+1}$ from the theorem 3.9, it follows that $U^*(n) = 2^3$. For $n \neq 3^{a+1}$, if $\sigma^*(k)|n = (2^m + 1)(2^p - 1)^a$, then the only possibility is $\sigma^*(k)|2^m + 1$. Therefore the result follows from the lemma 2.2.

Example 3.2. $U^*(35) = 2^2$, $U^*(51) = 2^4$, $U^*(7967) = 2^8$.

4. Conclusion

We study the maximum function $U^*(n)$ in detail and determine the exact value of $U^*(n)$ if n is prime power. There is also a scope for the study of the function $U^*(n)$ for other values of n.

Acknowledgement

We are grateful to the anonymous referee for reading the manuscript carefully and giving us many insightful comments.

Conflict of Interest

All authors declare no conflicts of interest in this paper.

References

1. C. Ashbacker, An introduction to the Smarandache function, Erhus Univ. Press ,Vail, AZ, 1995.
2. David M. Burton, Elementary number theory, Tata McGraw-Hill Sixth Edition, 2007.
3. E. Cohen, *Arithmetical functions associated with the unitary divisors of an integer*. Math. Zeits., **74** (1960), 66-80.
4. P. Moree, H. Roskam, *On an arithmetical function related to Euler's totient and the discriminator*. Fib. Quart., **33** (1995), 332-340.
5. J. Sandor, *On certain generalizations of the Smarandnche function*. Notes Num. Th. Discr. Math., **5** (1999), 41-51.
6. J. Sandor, *A note on two arithmetic functions*. Octogon Math. Mag., **8** (2000), 522-524.
7. J. Sandor, *On a dual of the Pseudo-Smarandache function.* Smarandnche Notition Journal, **13** (2002).
8. J. Sandor, *A note on exponential divisors and related arithmetic functions*. Sci. Magna, **1** (2005), 97-101.
9. J. Sandor, *The unitary totient minimum and maximum functions*. Sci. Studia Univ. "Babes-Bolyai", Mathematica, **2** (2005), 91-100.

10. J. Sandor, *The product of divisors minimum and maximum function*. Scientia Magna, **5** (2009), 13-18.

11. J. Sandor, *On the Euler minimum and maximum functions*. Notes Num. Th. Discr. Math., **15** (2009), 1-8.

AIMS Press

© 2017, Bhabesh Das, et al., licensee AIMS Press. This is an open access article distributed under the terms of the Creative Commons Attribution License (http://creativecommons.org/licenses/by/4.0)