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14 (MAT–1) 1016 (N/O)

2021

(Held in 2022)

MATHEMATICS

Paper : MAT–1016

(New and Old Course)

(Algebra)

Full Marks : 80

Time : Three hours

The figures in the margin indicate full marks for the questions.

1. Answer ***any four*** parts : 5×4=20

   (a) If G_1 and G_2 are two groups such that $H_1 \trianglelefteq G_1$ and $H_2 \trianglelefteq G_2$, then prove that

   (i) $H_1 \times H_2 \trianglelefteq G_1 \times G_2$

   (ii) $\frac{G_1 \times G_2}{H_1 \times H_2} \cong \frac{G_1}{H_1} \times \frac{G_2}{H_2}$

   2+3=5

Contd.

(b) Let G be a group and $H = \{(g, g) \mid g \in G\}$.

(i) Show that H is a subgroup of $G \times G$.

(ii) Prove that H is a normal subgroup of $G \times G$ if and only if G is Abelian.

2+3=5

(c) Define subnormal series of a group. When does a subnormal series become a normal series ?

Justify the statement "a subnormal series of a group may not be a normal series". 2+1+2=5

(d) Define a solvable group with a suitable example.

Show that S_n is not solvable for $n \geq 5$.

2+3=5

(e) Define a composition series of a group with a suitable example.

Justify the statement "Z has not composition series". 2+3=5

2. Answer ***any four*** parts : 5×4=20

(a) Define a PID. Prove that in a PID every irreducible element is prime.

(b) Let R be a PID. Prove that a proper ideal M of R is a maximal ideal of R if and only if it is generated by an irreducible element of R.

(c) Prove that in a PID, every pair of non-zero elements has an HCF and LCM.

(d) Define a UFD. Prove that every PID is a UFD.

(e) Consider the ring $\mathbb{R} = Z\left(\sqrt{-5}\right)$. Find the units of $\mathbb{R}$.

Prove or disprove "$\mathbb{R}$ is a PID".

3. Answer ***any four*** parts : 5×4=20

(a) If L is an algebraic extension of K and K is an algebraic extension of F, then show that L is an algebraic extension of F.

(b) Invalidate the statement "an algebraic extension is a finite extension".

(c) Find the splitting field and the degree of the splitting field of the following polynomials over Q :

(i) $x^2 - 2x + 1 \in Q[x]$

(ii) $x^2 - x + 1 \in Q[x]$

(d) Let F be a finite field of characteristic p, a prime, show that the number of elements q in F is a power of p.

(e) Show that a point (α, β) is constructible if and only if the real numbers α and β are constructible.

4. Answer ***any four*** parts : 5×4=20

(a) Prove that if V is an n–dimensional vector space over F and if $T \in A(V)$ has all its characteristic roots in F, then T satisfies a polynomial of degree n over F.

(b) Define index of nilpotency of $T \in A(V)$. Prove that if $T \in A(V)$ is nilpotent of index n_1, then a basis of V can be found such that the matrix of T in this basis has the form —

$$\begin{pmatrix} M_{n_1} & 0 & \cdots & 0 \\ 0 & M_{n_2} & \cdots & 0 \\ \vdots & & & \\ 0 & 0 & \cdots & M_{n_k} \end{pmatrix}$$

where $n_1 \geq n_2 \geq n_3 \geq \ldots \geq n_k$ and

$n_1 + n_2 + n_3 + \ldots + n_k = dim V$. 1+4=5

(c) Prove that if M, of dimension m, is cyclic with respect to T, then the dimension of $T^k(M)$ is $m-k$ for all $k \leq m$.

(d) Prove that two nilpotent linear transformations are similar if and only if they have the same invariants.

(e) Define a Jordan Block. Prove that if $T \in A_F(V)$ has all its distinct characteristic roots $\lambda_1, \lambda_2, \ldots\ldots \lambda_k$ in F, then a basis of V can be found in which the matrix T is of the form

$$\begin{pmatrix} J_1 & & & \\ & J_2 & & \\ & & \ddots & \\ & & & J_k \end{pmatrix}$$

where each J_i is

$$J_i = \begin{pmatrix} B_{i1} & & & \\ & B_{i2} & & \\ & & \ddots & \\ & & & B_{ir_1} \end{pmatrix}$$

and $B_{i1}, B_{i2}, \ldots, B_{ir_1}$ are basic Jordan blocks belonging to λ_i.
