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14 (MAT–1) 1046 (N/O)

2021

(Held in 2022)

MATHEMATICS

Paper : MAT–1046

(New and Old Course)

(Real Analysis and Lebesgue Measure)

Full Marks : 80

Time : Three hours

The figures in the margin indicate full marks for the questions.

1. Answer ***any four*** parts : 5×4=20

   (a) Test for uniform convergence of the sequence of function $\{f_n\}$, where

   $f_n(x) = \dfrac{n^2 x}{1+n^4 x^2}$ on [0, 1]. 5

Contd.

(b) Suppose $\{f_n\}$ is a sequence of functions, differentiable on $[a, b]$ and such that $\{f_n(x_0)\}$ converges for some point x_0 on $[a, b]$. If $\{f_n'\}$ converges uniformly on $[a, b]$, then prove that $\{f_n\}$ converges uniformly on $[a, b]$ to a function f and $f'(x) = \lim_{n \to \infty} f_n'(x), \ a \le x \le b$.

(c) Let f be a real valued continuous function defined on $[a, b]$. Prove that there exists a sequence of real polynomial $\{P_n\}$ which converges uniformly to f on $[a, b]$.

(d) Show that the series

$$\sum \frac{x}{(nx+1)\{(n-1)x+1\}}$$ is uniformly convergent on any interval $[a, b]$, $0 < a < b$, but only pointwise convergent on $[0, b]$.

(e) Find the radius of convergence and interval of convergence of the power series : 2+3=5

(i) $\sum \frac{2^n x^n}{n!}$

(ii) $\sum \frac{1.2.3. \ldots\ldots n}{1.3.5. \ldots\ldots (2n-1)} x^{2n}$

(f) (i) If a power series $\sum a_n x^n$ converges for $x = x_0$, then prove that it is absolutely convergent for every $x = x_1$ when $|x_1| < |x_0|$.

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(ii) Is the series $x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots$ converges absolutely on $(-1, 1)$?

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Answer ***any four*** parts : 5×4=20

(a) Give an example to show that a function of bounded variation may be continuous and conversely.

(b) If f is a real valued function on $[a, b]$, prove that of f is of bounded variation on $[a, b]$, then it is also of bounded variation on $[a, c]$ and $[c, b]$, where c is a point of $[a, b]$.

(c) Compute the positive, negative and total variation functions of

$f(x) = 3x^2 - 2x^3, -2 \le x \le 2.$

(d) Let $f:[a, b] \to \mathbb{R}^m$ be such that $f = (f_1, f_2, \ldots, f_m)$, each f_j is a real valued function. Prove that f is of bounded variation if and only if each component function f_j is of bounded variation on $[a, b]$. Hence verify whether the function $f(x) = (x, x^2)$ on $[0, 1]$ is of bounded variation or not.

3+2=5

(e) Let f and α be bounded functions on $[a, b]$. Prove that f is integrable with respect to α on $[a, b]$ if and only if for every $\varepsilon > 0$ there exists a partition P of $[a, b]$ such that

$U(P, f, \alpha) - L(P, f, \alpha) < \varepsilon$.

(symbols have usual meaning)

(f) Let f be monotonic and α be continuous on $[a, b]$, then prove that $f \in R(\alpha)$.

(symbols have usual meaning)

3. Answer ***any four*** parts : 5×4=20

(a) If $A_n, n = 1, 2, 3, ...$ are measurable subsets of $[a, b]$ and if $A_{n+1} \subseteq A_n$ for all n, then prove that $\bigcap_{n=1}^{\infty} A_n$ is measurable and

$$m\left(\bigcap_{n=1}^{\infty} A_n\right) = \lim_{n\to\infty} m(A_n).$$

(b) Construct an example to disprove that the sets of measure zero consist only of at most countably infinite number of points.

(c) If A is a measurable set, then show that $A + x = \{y + x : y \in A\}$ is measurable for each x and that the measures are the same.

(d) *(i)* Are continuous functions measurable ? Justify.

(ii) Does the existence of non-measurable sets imply that non-measurable functions also exist ? Justify.

(e) Prove that the function f on $[a, b]$ is measurable if and only if any one of the following two conditions holds :

(i) $\{x : f(x) > \alpha\}$ is measurable set for every $\alpha \in \mathbb{R}$.

(ii) $\{x : f(x) < \alpha\}$ is measurable set for every $\alpha \in \mathbb{R}$.

(f) Let f be a measurable function on $[a, b]$ and $k \in \mathbb{R}$. Prove that $|f|$ and kf are measurable.

4. Answer ***any four*** parts : 5×4=20

(a) Prove that every bound Riemann integrable function over $[a, b]$ is Lebesgue integrable and two integrals are equal.

Give an example to verify the converse of this result.

(b) Prove that if f is a bounded and Lebesgue integrable function on $[a, b]$ such that $f(x) = g(x)$ a.e. on $[a, b]$, where g is a bounded function on $[a, b]$, then g is Lebesgue integrable and

$$\int_a^b f\,dx = \int_a^b g\,dx .$$

(c) Define :

$$f(x)=\begin{cases}\dfrac{1}{x^{2/3}}, & 0<x\le 1\\ 0, & x=0\end{cases}$$

Show that f is Lebesgue integrable on $[0, 1]$ and $\int_0^1 \frac{dx}{x^{2/3}}=3$, and find $F(x, 2)$.

(d) Let $\{f_n\}$ be a sequence of measurable functions on a measurable subset $A \subseteq [a, b]$ such that $\lim_{n\to\infty} f_n(x)=f(x)$. If there exists a constant M such that $|f_n(x)|\le M$ for all x and n, then prove that $\lim_{n\to\infty}\int_A f_n(x)\,dx=\int_A f(x)\,dx$.

(e) Let $\{f_n\}$ be a sequence of non-negative measurable functions on $[a, b]$. Let $\lim_{n\to\infty} f_n(x)=f(x)$ a.e. on $[a, b]$. Prove that $\lim_{n\to\infty}\int_a^b f_n(x)\,dx \ge \int_a^b f(x)\,dx$, if f is Lebesgue integrable on $[a, b]$, otherwise $\lim_{n\to\infty}\int_a^b f_n(x)\,dx=\infty$.

(f) Justify that on the set of finite measure, uniformly convergent sequences of bounded functions are boundedly convergent.
