UG/Sem-1/MAT-HC2

2022

Mathematics (Honours)

Paper : MAT-HC-1026

(Algebra)

Time : 3 hours *Full Marks : 80*

The figures in the margin indicate full marks for the questions.

1. Answer the following questions: $1\times 10=10$

(a) Write the negation of the statement:

$\forall a \in R(\exists x \in A(x \geq a))$

(b) Let $A=\{(x,y)\in R^2 : x+y=0\}$ and $B=\{(x,y)\in R^2 : xy=1\}$. Find $A \cap B$.

(c) Write true or false:

The circle $\{(x,y)\in R^2 : x^2+y^2=1\}$ is a graph of any function.

(d) For $z_1, z_2 \in C$, is the number $z_1\bar{z}_2+\bar{z}_1 z_2$ a real number ?

(e) For what values of h and k, the following system is consistent?

$$2x_1 - x_2 = h$$

$$-6x_1 + 3x_2 = k$$

(f) Write true or false:

An inconsistent system has more than one solution.

(g) Is the function $f : Z \to Z$ defined by $f(x)=2x+5$ one-one?

(h) Write true or false:

If a set $S = \{x_1, x_2, x_3, \ldots\ldots\ldots\ldots, x_p\}$ in R^n contains a zero vector, then the set is linearly independent.

(i) If $A = \begin{bmatrix} 6 & 1 \\ 3 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 4 & 3 \\ 1 & 2 \end{bmatrix}$, then find $\det(A - B)$.

(j) Find the polar coordinates of the point $(\sqrt{3}, 1)$.

2. Answer the following questions: 2×5=10

(a) Find the polar representation of the complex number $z = 1 + \cos\alpha + i\sin\alpha$, $\alpha \in (0, 2\pi)$

(b) If n divides k then show that any root of $z^n - 1 = 0$ is a root of $z^k - 1 = 0$.

(c) Prove that for an integer n, if $n^3 - 1 = 0$ is even, then n is odd using contrapositive implication.

(d) Show that the vectors $v_1 = (2, 2, -3)$, $v_2 = (0, -4, 1)$, $v_3 = (3, 1, -4)$ are linearly dependent.

(e) Find the geometric image of the complex number z in

$$|z - 1| = 3.$$

3. Answer ***any four*** questions of the following: 5×4=20

(a) If $z + \dfrac{1}{z} = \sqrt{3}$ then find the value of $z^n + \dfrac{1}{z^n}$. 5

(b) Prove that there are infinitely many prime numbers. 5

(c) Show that a linear transformation $T: V \to V$ on a vector space V is one-to-one if and only if T is onto. 5

(d) Let $f: X \to Y$ be a mapping and $B_1, B_2 \subset Y$. Prove that

$$f^{-1}(B_1 \cap B_2) = f^{-1}(B_1) \cap f^{-1}(B_2)$$ 5

(e) (a) Examine if the following system of equation is consistent or not. 5

$$x_1 + 3x_3 = 2$$

$$x_2 - 3x_4 = 3$$

$$-2x_2 + 3x_3 + 2x_4 = 1$$

$$3x_1 + 7x_4 = -5$$

(f) Find the inverse of the matrix $A = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & 3 \\ 4 & -3 & 8 \end{bmatrix}$ if it exists by performing suitable row operations on the augmented matrix $[A : I]$. 5

4. Answer ***any four*** questions of the following: 10×4=40

(a) (i) Prove that De Moivre's theorem holds for negative integer exponents. 3

(ii) Let p be a prime number and let $\varepsilon = \cos\frac{2\pi}{p} + i\sin\frac{2\pi}{p}$, show that if $a_0, a_1, \ldots\ldots, a_{p-1}$ are non zero integers, the relation

$a_0 + a_1\varepsilon + ... + a_{p-1}\varepsilon^{p-1} = 0$ holds if and only if $a_0 = a_1 = ... = a_{p-1}$. 3

(iii) Show that every integer greater than 1 has a prime divisor. 4

(b) (i) If $T : X \to Y$ is a bijection function then prove that $f^{-1}of = I_X$ and $fof^{-1} = I_Y$, where I_X and I_Y are identity functions on X and Y respectively. 5

(ii) If $f : X \to Y$ and $g : Y \to Z$ are bijection functions, then show that gof is a bijection function and $(gof)^{-1} = f^{-1}og^{-1}$.

5

(c) Define equivalence relation on a non-empty set X. Show that the relation congruence modulo n, where $n \neq 0$, is any fixed integer on the set Z of integers, defined by

$$a \equiv b(\operatorname{mod} n) iff\, n \mid a - b$$

is an equivalence relation. Find all the distinct equivalence classes of Z if $n=4$, so that Z is the union of these. 1+4+5=10

(d) Define well-ordering principle. Prove that if $a, b \in Z$ with $a \in N$, then there exists unique integers q and r such that-

(i) $b = aq + r$,

(ii) $0 \leq r < a$. 2+8=10

(e) (i) Let $T : U \to V$ be a linear transformation. Show that $KerT = \{0\}$ if and only if T is one-one 3

(ii) Consider the mapping $T: R^3 \to R$ such that

$T(x_1, x_2, x_3) = x_1^{\ 2} + x_2^{\ 2} + x_3^{\ 2}$. Examine the linearity of T. 3

(iii) If $T: R^3 \to R^3$ such that

$T(x, y, z) = (2x - 3y + 4z, 5x - y + 2z, 4x + 7y)$.

Find the matrix of T with respect to the usual basis of R^3. 4

(f) (i) Let $X = N \times N$, where N is the set of positive integers. Define a relation $\sim$ on X as $(a, b) \sim (c, d)$ if and only if $a + d = b + c$. Show that $\sim$ is an equivalence relation on X. 4

(ii) Reduce matrix A to echelon form by row reduce method and locate the pivot columns of A, where

$$A = \begin{bmatrix} 0 & -3 & -6 & 4 & 9 \\ -1 & -2 & -1 & 3 & 1 \\ -2 & -3 & 0 & 3 & -1 \\ 1 & 4 & 5 & -9 & -7 \end{bmatrix}$$ 6

■■■